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The symplectic camel and phase space quantization

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Abstract
We show that a result of symplectic topology, Gromov's non-squeezing theorem, also known as the 'principle of the symplectic camel', can be used to quantize phase space in cells. That quantization scheme leads to the correct energy levels for integrable systems and to Maslov quantization of Lagrangian manifolds by purely topological arguments. We finally show that the argument leading to the proof of the non-squeezing theorem leads to a classical form of Heisenberg's inequalities.

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1. Introduction

The common conception of Liouville's theorem is that under a Hamiltonian flow a volume in phase space can be made as thin as one likes provided the volume remains constant (see [6]; also [18]). Thus, it would be possible to pass the proverbial camel through the eye of a needle no matter how small the eye! This is in fact not true for the 'symplectic camel'. For a given Hamiltonian process in phase space, it is *not* possible to shrink a cross-section defined by conjugate coordinates like x and p_x to zero. In other words, we have a minimum cross-sectional area within a given volume that cannot be shrunk further: it is as if the uncertainty principle had left a 'footprint' in classical mechanics—or, perhaps, as if the symplectic camel had left a footprint in quantum mechanics .... This is indeed a very surprising phenomenon, especially since it ceases to hold if one replaces the x, p_x plane by any other plane of non-conjugate variables (e.g., x, p_y). It is in fact a consequence of a deep theorem from symplectic topology, which was proved by Gromov [11] in the mid 1980s, and called the 'non-squeezing' theorem, or the 'principle of the symplectic camel'.

This principle—which seems not to be widely known outside a few specialized mathematical circles—highlights the deep difference between volume preservation and symplectic invariance.

The aim of this paper is to show that the non-squeezing theorem leads can be used to mathematically justify a quantization of phase space in 'cells' $\mathcal{C}(h)$ on the boundary of which every periodic orbit has exactly action $\frac{1}{2}h$. These cells can be *unbounded* and infinite *volume*, which makes them totally different from the usual 'quantum cells' used in thermodynamics or quantum chemistry. We will see that this particular quantization scheme leads, when applied to the completely integrable systems of mechanics, to the correct semi-classical ground energy levels (and thus to the exact quantization of the harmonic oscillator).

In section 2 we review the non-squeezing theorem and examine some of its mathematical consequences; we also define the related notion of *symplectic capacity*;

In section 3 we briefly discuss the deep relationship between periodic orbits in phase space and the symplectic capacity of convex subsets.

In section 4 we quantize phase space in cells: quantum cells are convex (possibly unbounded) subsets of phase space, having symplectic capacity $\frac{h}{2}$, and postulate that there can be no periodic orbits inside such a cell in quantum mechanics. This leads us to the correct ground-energy levels for the n-dimensional anisotropic harmonic oscillator.

In section 5 we show how the notion of quantum cell leads to the Keller–Maslov quantization of Lagrangian manifolds. The argument we use is purely topological, i.e. we do not need to invoke, as in the usual approaches, the WKB approximation.

In section 6, we show, using the techniques leading to the proof of the non-squeezing theorem, that there is a classical form of Heisenberg's inequalities. (That there is a deep relation between the symplectic camel and the uncertainty principle was already noted and discussed by Viterbo in [24].)

We conclude by discussing some open questions and possible extensions.

2. The non-squeezing theorem

We begin by stating Gromov's theorem precisely. We will use the generalized coordinates $x = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$. Consider the phase space ball

$$B(R) = \left\{(x, p) : |x|^2 + |p|^2 \leqslant R^2\right\}$$

and the 'symplectic cylinder'

$$Z_j(r) = \left\{(x, p) : x_j^2 + p_j^2 \leqslant r^2\right\}$$

($1 \leqslant j \leqslant n$) based on the x_j, p_j plane. Recall that a symplectomorphism (or canonical transformation) is a diffeomorphism of phase space whose Jacobian matrix is symplectic at every point.

Theorem 1 (Gromov). *There exists a symplectomorphism f of $\mathbb{R}^n_x \times \mathbb{R}^n_p$ sending $B(R)$ inside $Z_j(r)$ if and only if $R \leqslant r$:*

$$f(B(R)) \subset Z_j(r) \Longleftrightarrow R \leqslant r.$$

It is essential for the validity of the theorem to assume that $Z_j(r)$ is a symplectic cylinder in theorem 1. Here is a counter-example: every symplectic mapping $m_\lambda : (x, p) \longmapsto (\lambda x, p/\lambda)$ sends $B(R)$ into a cylinder $\{(x, p) : x_1^2 + p_2^2 \leqslant r\}$ for all R provided that $\lambda \leqslant R/r$. On the other hand, it is always possible to squeeze the ball $B(R)$ inside a symplectic cylinder $Z_j(r)$ if one uses general volume-preserving diffeomorphisms.

No 'easy' proofs of Gromov's theorem are known, and we refer to Gromov's original paper [11], or to Hofer–Zehnder [12]. (Viterbo [23] has given an interesting alternative proof of Gromov's theorem based on the notion of generating function.) Here is, however, a semi-heuristic justification of this result. We give it for $n = 2$, but the argument goes through in an arbitrary number of dimensions. Consider the isotropic two-dimensional harmonic oscillator with Hamiltonian

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right).$$

The solutions of the associated Hamilton equations are the 2π-periodic functions:

$$x = x' \cos t + p_x' \sin t \qquad y = y' \cos t + p_y' \sin t$$
$$p_x = -x' \sin t + p_x' \cos t \qquad p_y = -y' \sin t + p_y' \cos t.$$

Let us now fix the initial point (x', y', p_x', p_y') on the sphere $S(R)$ with radius R in $\mathbb{R}^2_{x,y} \times \mathbb{R}^n_{p_x,p_y}$, centred at the origin. Since H is a constant of the motion we will have

$$p_x^2 + p_y^2 + x^2 + y^2 = R^2$$

for all times, so that the orbit will stay forever on the sphere (all such orbits are in fact the big circles of the sphere). For one period, the action of such an orbit is πR^2. Set now $x'^2 + p_x'^2 = r^2$; then the initial point (x', y', p_x', p_y') is on the cylinder

$$C(r) = \left\{(x, y, p_x, p_y) : x^2 + p_x^2 = r^2\right\}$$

and the whole orbit lies on that cylinder, and winds around it; the action for a period is this time πr^2 (notice that if we had chosen instead a cylinder based on the plane x, y, then the orbits would have been straight lines, and hence not periodic). Suppose now that we deform the sphere $S(R)$, using symplectomorphisms (for instance, a Hamiltonian flow), so that it 'fits exactly' inside the cylinder $C(r)$, touching it along a circle. Action being a symplectic invariant (see e.g. [1]), we must have $\pi R^2 = \pi r^2$, so that $R = r$.

Gromov's theorem implies (and is, in fact, equivalent to (see [9])):

Theorem 2. *Let* Pr_j *be the projection of phase space on the symplectic plane* $\mathbb{R}_{x_j} \times \mathbb{R}_{p_j}$*. Then, for every symplectomorphism* f*, we have*

$$\text{Area}\,\mathrm{Pr}_j\, f(B(R)) \geqslant \pi R^2. \tag{1}$$

Proof. The 'shadow' $\mathrm{Pr}_j\, f(B(R))$ is a compact and connected submanifold of $\mathbb{R}_{x_j} \times \mathbb{R}_{p_j}$ with smooth boundary γ. Setting

$$\text{Area}(\mathrm{Pr}_j\, f(B(R))) = \pi r^2.$$

$\mathrm{Pr}_j\, f(B(R))$ is diffeomorphic to the disc $D_j(r) : x_j^2 + p_j^2 \leqslant r^2$. Let Φ be a volume-preserving diffeomorphism $\Phi : \mathrm{Pr}_j\, f(B(R)) \longrightarrow D_j(R)$ (the existence of such a Φ is guaranteed by a famous theorem of Moser [4, 17]). Define now a diffeomorphism g of $\mathbb{R}^n_x \times \mathbb{R}^n_p$ by $g(x', p') = (x, p)$ where

$$x_k = x_k' \qquad p_k = p_k' \qquad \text{if} \quad k \neq j$$
$$(x_j, p_j) = \Phi(x_j', p_j').$$

Since Φ is area preserving we have $\mathrm{d}p_j \wedge \mathrm{d}x_j = \mathrm{d}p_j' \wedge \mathrm{d}x_j'$ so that g is in fact a symplectomorphism. Let now $T(r)$ be the cylinder orthogonal to the x_j, p_j plane and whose generators pass through the boundary γ of $\mathrm{Pr}_j\, f(B(R))$: $T(r)$ is thus a cylinder in phase space containing $f(B(R))$, and this cylinder is transformed into $Z_j(r)$ by the symplectomorphism g. But then

$$g(f(B(R))) \subset Z_j(r)$$

so that we must have $R \leqslant r$ in view of Gromov's theorem; the inequality (1) follows. □

Let $\mathcal{D}$ be a subset of phase space $\mathbb{R}^n_x \times \mathbb{R}^n_p$. We will call *symplectic radius* of $\mathcal{D}$ the supremum of all r such that we can send the phase space ball $B(r)$ inside $\mathcal{D}$ using a symplectomorphism. We will call the *symplectic capacity* of $\mathcal{D}$, and denote by $\mathrm{Cap}(\mathcal{D})$ the number πr^2. (The notion of symplectic capacity was first introduced by Ekeland and Hofer in [5]; there are other non-equivalent definitions of symplectic capacities (see [12, 14]).) With these notations, Gromov's theorem can be restated as

$$B(R) \subset \mathcal{D} \subset Z_j(R) \Longrightarrow \mathrm{Cap}(\mathcal{D}) = \pi R^2$$

which shows that sets of very different shapes and volumes can have the same capacity. Also observe that the capacity of a ball $B(R)$, or of a symplectic cylinder, is independent of the dimension of the phase space, as it is always πR^2. Since the ball $B(R)$ in $\mathbb{R}^n_x \times \mathbb{R}^n_p$ has volume $\pi^n R^{2n}/n!$ we thus have

$$Vol\, B(R) = \frac{1}{n!}\left[\mathrm{Cap}(B(R))\right]^n. \tag{2}$$

In general, $0 \leqslant \mathrm{Cap}(\mathcal{D}) \leqslant +\infty$, however, $0 < \mathrm{Cap}(\mathcal{D}) < +\infty$ for all non-empty open bounded sets $\mathcal{D}$: translating if necessary $\mathcal{D}$, we can namely always find r and R such that $B(r) \subset \mathcal{D} \subset B(R)$, and hence

$$\pi r^2 \leqslant \mathrm{Cap}(\mathcal{D}) \leqslant \pi R^2.$$

Here we used implicitly the fact that the symplectic capacity is translation invariant; in fact it follows from its definition that $\mathrm{Cap}(f(\mathcal{D})) = \mathrm{Cap}(\mathcal{D})$ for every symplectomorphism f. Let us finally consider, for further use the ellipsoid

$$\mathcal{E} : \sum_{j=1}^{n} \frac{1}{R_j^2}\left(p_j^2 + x_j^2\right) \leqslant 1$$

where $R_1 < \cdots < R_n$. (One can show that the equation of any ellipsoid in phase space can be put in this 'normal form'.) The sequence $R = (R_1, \ldots, R_n)$ is called the *symplectic spectrum* of $\mathcal{E}$; one proves that (see [12])

$$\mathrm{Cap}\,\mathcal{E} = \pi R_1^2. \tag{3}$$

Let us next compare capacity and volume. In the case $n = 1$ the symplectic capacity as defined above is just the area

$$\mathrm{Cap}(\mathcal{D}) = \left| \int_{\mathcal{D}} \mathrm{d}p\, \mathrm{d}x \right|$$

(for the Hofer–Zehnder capacity on symplectic manifolds defined in [12] this result is much less trivial; see [21], or [12], pp 100–103). That property does not extend to higher dimensions: if $n > 1$, $(Vol(\mathcal{D}))^{1/n}$ is certainly not a symplectic capacity on $\mathbb{R}^n_x \times \mathbb{R}^n_p$, because $Vol\, Z_j(R) = +\infty$.

3. Periodic orbits

Let H be a Hamiltonian function on phase space $\mathbb{R}^n_x \times \mathbb{R}^n_p$, and

$$X_H = (\nabla_p H, -\nabla_x H)$$

the associated Hamilton vector field. We will assume that H is time independent, although this is by no means an essential restriction for the validity of our results. We will call 'energy shell' a non-empty level set of the Hamiltonian H. We will always denote an energy shell by the symbol ∂M, whether it is the boundary of a set M or not. Thus

$$\partial M = \left\{(x, p) \in \mathbb{R}^n_x \times \mathbb{R}^n_p : H(x, p) = E\right\}.$$

We notice that any smooth hypersurface of phase space is the energy shell of some Hamiltonian function H: it suffices to choose for H any smooth function on $\mathbb{R}^n_x \times \mathbb{R}^n_p$ and keeping some constant value E on ∂M. It is a remarkable result (well known from the regularization theory of collision singularities in Kepler's two-body problem) that Hamiltonian periodic orbits on a hypersurface are independent of the choice of the Hamiltonian having that hypersurface as an energy shell. Periodic orbits are thus intrinsically attached to any hypersurface in phase space: let H and K be two Hamiltonians, and suppose that there exist two constants h and k such that

$$\partial M = \{(x, p) : H(x, p) = h\} = \{(x, p) : K(x, p) = k\} \tag{4}$$

with $\nabla_{(x,p)} H \neq 0$ and $\nabla_{(x,p)} K \neq 0$ on ∂M. Then the Hamiltonian vector fields X_H and X_K have the same periodic orbits on ∂M (see [12], chapter 1). In view of this result, in what follows we will talk about the 'periodic orbits of a set ∂M' without singling out a particular Hamiltonian.

The problem of the existence of periodic orbits on a given energy shell ∂M is a very difficult one, which has not yet been completely solved. One of the oldest in that direction is due to Seifert [20] in 1948: he showed that every *compact* energy shell for a Hamiltonian

$$H = \frac{p^2}{2m} + U$$

contains at least one periodic orbit, provided it is homeomorphic to a convex set. The following general result is due to Rabinowitz [19] (also see [25]):

Proposition 3. *If the boundary of a compact and convex region in phase space is C^2, then it carries at least one periodic orbit.*

It turns out that there is a fundamental relation between symplectic capacity and the action of periodic orbits in phase space.

Theorem 4. *Let M be a compact and convex region in phase space. Then every periodic orbit γ on ∂M is such that*

$$\left| \int_\gamma p \, \mathrm{d}x \right| \geqslant \mathrm{Cap}(M) \tag{5}$$

and there exists at least one periodic orbit γ_0 for which we have the equality

$$\left| \int_{\gamma_0} p \, \mathrm{d}x \right| = \mathrm{Cap}(M).$$

See [12] for a proof of this theorem.

4. Quantized phase space cells

In semi-classical physics and quantum chemistry it is common usage to invoke Heisenberg's inequalities to partition phase space in 'cells' with volume having an order of magnitude h^3. We propose here a quantization scheme based on the property of the symplectic camel. It consists in postulating that no periodic orbits exist, in quantum mechanics, on subsets of phase space with symplectic capacity smaller than $\frac{1}{2}h$. This postulate leads to the Keller–Maslov quantization of Lagrangian manifolds (and, in particular, to the correct ground level energies for integrable systems)

The fact that the lowest energy levels are different from zero is often motivated in the physical literature by saying that an observed quantal harmonic oscillator cannot be at rest (that is, one cannot find $x = 0$ and $p = 0$), because this would violate Heisenberg's uncertainty principle. We will see that the non-zero ground energy levels actually have a topological origin, and can be viewed a consequence of the principle of the symplectic camel.

Definition 5. *A* quantum cell *is a convex subset $\mathcal{C}(h)$ of phase space with capacity $\frac{1}{2}h$:*

$$\operatorname{Cap}\mathcal{C}(h) = \tfrac{1}{2}h.$$

Note that we *do not* require that $\mathcal{C}(h)$ be compact: a ball with radius $\sqrt{\hbar}$ is a quantum cell, but so is also any symplectic cylinder with radius $\sqrt{\hbar}$. Quantum cells can thus be *unbounded*, and have *infinite volume*. When a cell in $2n$-dimensional phase space is the ball $B(\sqrt{\hbar})$, then its volume is

$$Vol\, B\big(\sqrt{\hbar}\big) = \frac{h^n}{2^n n!}$$

in view of (2). When $n = 3$, corresponding to the case of the phase space of a single particle, the volume of such a cell is thus $h^3/48$, and for two particles ($n = 6$) it is $h^6/46\,080$.

We now make the following *physical* assumption:

Postulate 1. *In quantum mechanics the only admissible energy shells are those with capacity $\geqslant \frac{1}{2}h$.*

I view of the 'capacity = action' result of theorem 4, this postulate is immediately equivalent to:

Postulate 2. *In quantum mechanics, the action of any Hamiltonian periodic orbit is at least $\frac{1}{2}h$, and a periodic orbit can have action $\frac{1}{2}h$.*

We will call an orbit γ_0 for which equality occurs a *minimal periodic orbit*:

$$\oint_{\gamma_0} p\,\mathrm{d}x = \tfrac{1}{2}h.$$

We are going to see that the minimum capacity/action principle suffices to determine the ground energy levels for the harmonic oscillator in arbitrary dimension n.

Proposition 6. *Consider the n-dimensional harmonic oscillator with Hamiltonian*

$$H = \sum_{j=1}^{n} \frac{1}{2m_j}\big(p_j^2 + m_j\omega_j x_j^2\big). \tag{6}$$

The minimum capacity principle implies that the ground energy level of that Hamiltonian is

$$E_0 = \sum_{j=1}^{n} \tfrac{1}{2}\hbar\omega_j. \tag{7}$$

Proof. First perform the change of variables

$$(x, p) \longmapsto (Lx, L^{-1}p)$$

where L is the $n \times n$ diagonal matrix with diagonal entries $(m_j\omega_j)^{-1/2}$; this changes H into

$$H' = \sum_{j=1}^{n} \frac{\omega_j}{2}\left(p_j^2 + x_j^2\right).$$

The change of variables above being symplectic, this transformation does not affect the action form $p\,\mathrm{d}x$, and it does not change the symplectic capacities of sets; we may therefore prove the result with the Hamiltonian H replaced by H'. We then remark that each orbit

$$\gamma : \begin{cases} x_1 = x_1' \cos\omega_1 t + p_1' \sin\omega_1 t & p_1 = x_1' \sin\omega_1 t - p_1' \cos\omega_1 t \\ \vdots & \\ x_n = x_n' \cos\omega_n t + p_n' \sin\omega_n t & p_n = x_n' \sin\omega_n t - p_n' \cos\omega_n t \end{cases}$$

lies not only on the ellipsoid which is the energy shell of the Hamiltonian H', but also on each of the symplectic cylinders

$$Z_j(R_j) = \left\{(x, p) : x_j^2 + p_j^2 = R_j^2\right\}$$

with $R_j^2 = x_j'^2 + p_j'^2$ and $1 \leqslant j \leqslant n$. These cylinders carry periodic orbits, and their capacities must satisfy the conditions

$$\mathrm{Cap} Z_j(R_j) = \pi R_j^2 \geqslant \tfrac{1}{2}h$$

in view of postulate 1. If γ_0 is a minimal periodic orbit, it will thus satisfy

$$E(\gamma_0) = \sum_{j=1}^{n} \tfrac{1}{2}\omega_j R_j^2 = \sum_{j=1}^{n} \tfrac{1}{2}\hbar\omega_j$$

which is the result predicted by standard quantum mechanics. □

Remark 7. The crucial point in the proof of proposition 6 was that a symplectic cylinder, albeit unbounded, has the same capacity as the ball to which it is tangent along a circle. If we had only considered bounded cells (balls or ellipsoids) as objects to 'quantize', then the argument would have led us to

$$E(\gamma_0) = \tfrac{1}{2}\hbar \sup_j \omega_j$$

which is wrong, even when the oscillator is isotropic.

5. Quantization of integrable systems

We now consider a completely integrable system with Hamiltonian H. There are n thus independent constants of the motion $F_1 = H, F_2, \ldots, F_n$ in involution: $\{F_j, F_k\} = 0$. It is well known (see, e.g., [1]) that given an energy shell $H = E$, through every point $z_0 = (x_0, p_0)$ of that energy shell passes a Lagrangian manifold V carrying the orbits passing through z_0. Moreover, when V is connected (which we assume from now on) there exists a symplectomorphism

$$f : V \longrightarrow (S^1)^k \times \mathbb{R}^{n-k} \tag{8}$$

where $(S^1)^k$ is the product of k unit circles, each lying in some coordinate plane x_j, p_j. In particular, if V is compact then it is symplectomorphic to the torus $T^n = (S^1)^n$.

Now, the minimum capacity/action principle imposes a condition on the energy shells of any Hamiltonian. That condition is that there should be no periodic orbits with action less than $\frac{1}{2}h$, and that there should exist 'minimal periodic orbits' having precisely $\frac{1}{2}h$ as action. In fact, we have the following result which ties the minimum capacity/action principle to the Maslov index of loops, and thus justifies the 'EBK' or 'Bohr–Sommerfeld' quantization condition by a purely *topological* argument:

Theorem 8. *Let V be a Lagrangian manifold associated with a Liouville integrable Hamiltonian H and carrying minimal action periodic orbits. Then we have*

$$\frac{1}{2\pi\hbar}\int_\gamma p\,\mathrm{d}x - \frac{1}{4}m(\gamma) = 0 \tag{9}$$

for every loop on V.

Proof. Since the actions of loops are symplectic invariants, we can use the symplectomorphism (8) to reduce the proof to the case $V = (S^1)^k \times \mathbb{R}^{n-k}$. Since the first homotopy group of V is

$$\pi_1((S^1)^k \times \mathbb{R}^{n-k}) \equiv \pi_1(S^1)^k \equiv (\mathbb{Z}^k, +)$$

it follows that every loop in V is homotopic to a loop of the type

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_k(t), 0, \ldots, 0) \qquad 0 \leqslant t \leqslant T$$

where γ_j are loops on S^1: $\gamma_j(0) = \gamma_k(T)$. On the other hand, every loop on S^1 is homotopic to a loop

$$\varepsilon_j(t) = (\cos\omega_j t, \sin\omega_j t) \qquad 0 \leqslant t \leqslant T_j$$

so there must exist positive integers μ_j $(1 \leqslant j \leqslant n)$ such that

$$\mu_1 T_1 = \cdots = \mu_k T_k = T.$$

(In particular, the frequencies $\omega_1, \ldots, \omega_k$ must be commensurate, i.e. $\omega_i : \omega_j$ is rational for all i and j.) We can thus identify γ_j with $m_j \varepsilon_j$, the loop ε_j described 'μ_j times':

$$\mu_j \varepsilon_j(t) = (\cos\omega_j t, \sin\omega_j t) \qquad 0 \leqslant t \leqslant T$$

and it follows that any loop in $V = (S^1)^k \times \mathbb{R}^{n-k}$ is homotopic to a loop

$$\gamma = \mu_1\varepsilon_1 + \cdots + \mu_k\varepsilon_k.$$

We thus have

$$\oint_\gamma p\,\mathrm{d}x = \sum_{j=1}^{k} \mu_j \oint_{\varepsilon_j} p_j\,\mathrm{d}x_j$$

and using the same argument as that leading to the proof of formula (7) in proposition 6, we must have

$$\oint_{\gamma_j} p_j\,\mathrm{d}x_j = \tfrac{1}{2}h \qquad (1 \leqslant j \leqslant k)$$

and hence

$$\oint_\gamma p\,\mathrm{d}x = \tfrac{1}{2}\left(\sum_{j=1}^{k} \mu_j\right)h.$$

Now, the Maslov index of such a loop γ in $(S^1)^k \times \mathbb{R}^{n-k}$ is by definition

$$m(\gamma) = 2\sum_{j=1}^{k} \mu_j$$

(see for instance [3, 9]) hence the Keller–Maslov condition (9). □

Remark 9. We urge the reader to note that the quantization condition (9) is about *loops* in the Lagrangian manifold, and not about periodic orbits! The quantization condition is independent of the existence of periodic orbits on the Lagrangian manifold, and thus applies, in particular, to the case of the n-dimensional harmonic oscillator with incommensurate frequencies.

The important result here above motivates the following definition, which was arrived at by other means by Maslov [15, 16], following previous trail-blazing work of Keller [13]:

Definition 10 (Keller–Maslov). *A Lagrangian manifold V is said to satisfy the Keller–Maslov quantization condition, or to be a quantized Lagrangian manifold, if*

$$\frac{1}{2\pi\hbar}\int_{\gamma} p\,\mathrm{d}x - \frac{1}{4}m(\gamma) \quad \textit{is an integer} \tag{10}$$

for every loop γ in V.

One easily verifies, by a direct calculation, that in the case of the n-dimensional anisotropic harmonic oscillator with Hamiltonian (6), the Lagrangian manifolds singled out by the 'selection rule' (10) are precisely those on which the energy is given by

$$E_{N_1,\ldots,N_n} = \sum_{j=1}^{n}\left(N_j + \tfrac{1}{2}\right)\hbar\omega_j$$

which are the correct values predicted by quantum mechanics. For more general Hamiltonians, condition (10) does not in general yield the correct energy levels.

Remark 11. The step from the ground state to a general excited state involves replacing zero in the right-hand side of equation (9) by an integer (in equation (10)). While it would be interesting to give a topological interpretation of this step in terms of Gromov's theorem, it can be explained within the framework of semiclassical mechanics in phase space (see [7–9]). The argument is the following (we restrict ourselves to the one-dimensional harmonic oscillator): one seeks to define stationary 'phase space wavefunctions' of the type

$$\Psi(\theta) = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(r,\theta)}a(r,\theta)\sqrt{r\,\mathrm{d}\theta}$$

on the phase space trajectories. Here θ is the polar angle, r the radius, $a(r,\theta)$ a 2π-periodic function of θ and $\varphi(r,\theta)$ is the action

$$\varphi(r,\theta) = \int_0^{\theta} p(\theta)\,\mathrm{d}r\,(\theta) = \frac{r^2}{2}(\sin\theta\cos\theta - \theta).$$

The argument of $\mathrm{d}\theta$ is defined by $\arg\,\mathrm{d}\theta = m(\theta)\pi$ where $m(\theta) = [\theta/2\pi]$ ($[\cdot]$ the integer part function). The square root $\sqrt{r\,\mathrm{d}\theta}$ is thus

$$\sqrt{r\,\mathrm{d}\theta} = \mathrm{i}^{[\theta/2\pi]\pi}\sqrt{|r\,\mathrm{d}\theta|}.$$

Now, if the quantization condition (9) is satisfied we have $\Psi(\theta + 2\pi) = \Psi(\theta)$ so the 'wavefunction' Ψ is defined on the trajectory itself. However, while (9) is sufficient for $\Psi(\theta + 2\pi) = \Psi(\theta)$ to hold, it is not a necessary condition: we will actually have $\Psi(\theta + 2\pi) = \Psi(\theta)$ on *all* trajectories satisfying the Keller–Maslov condition (10).

5.1. The uncertainty principle in classical mechanics

Consider now a point in phase space $\mathbb{R}^n_x \times \mathbb{R}^n_p$, and suppose that by making position and momentum measurements we are able to find out that this point lies in a ball B with radius R. Then the 'range of uncertainty' in our knowledge of the values of a pair (x_j, p_k) of position and momentum coordinates lies in the projection of that ball on the x_j, p_k plane. Since this projection is a circle with area πR^2, one might thus say that πR^2 is a lower bound for the uncertainty range of joint measurements of x_j and p_k. Suppose now that the system moves under the influence of a Hamiltonian flow (f_t). The ball B will in general be distorted by the flow into a more or less complicated region of phase space, while keeping the same volume. Since conservation of volume does not imply conservation of shape, a first guess is that one can say nothing about the time-evolution of the uncertainty range of (x_j, p_k), which can *a priori* become arbitrarily small. This guess is, however, wrong because of Gromov's theorem: as

B is getting distorted by the Hamiltonian flow (f_t), the projection $\mathrm{Pr}_j(f_t(B))$ of $f_t(B)$ on each *conjugate* variable plane x_j, p_j will, however, never shrink, and always have an area superior or equal to πR^2. This is contrast with the areas of the projections of $f_t(B)$ onto the non-conjugate planes x_j, p_k $(j \neq k)$, which can take arbitrarily small values. One can thus say that the uncertainty range of every pair (x_j, p_j) of conjugate variables can never be decreased by Hamiltonian motion, and this property can of course be viewed as a *classical* topological form of Heisenberg's uncertainty principle. Let us quantify the argument above. Assume H, for simplicity, quadratic. We denote by X_i' and P_j' the stochastic variables whose values are the results of the measurements, at initial time $t' = 0$, of the ith position and jth momentum coordinate, respectively. We assume that these variables are independent. Let $\Delta x_i' = \sigma(X_i')$ and $\Delta p_j' = \sigma(P_j')$ be the standard deviations at time t'; and Δx_i, Δp_j those at time t. We ask the following question:

> *What happens to Δx_i and Δp_j during the motion? More precisely what can we predict about Δx_i and Δp_j, knowing $\Delta x_1', \ldots, \Delta x_n'$, $\Delta p_1', \ldots, \Delta p_n'$?*

We claim that if the motion is governed by a quadratic Hamiltonian function, then following result holds:

Proposition 12. *Suppose that we have $\Delta p_j' \Delta x_j' \geqslant \varepsilon$ for $1 \leqslant j \leqslant n$. Then we also have $\Delta p_j \Delta x_j \geqslant \varepsilon$ for $1 \leqslant j \leqslant n$ and for all times t.*

Proof. Since the Hamiltonian H is quadratic, the flow consists of symplectic matrices; writing

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x' \\ p' \end{pmatrix} \tag{11}$$

the coordinates x_j, p_j are given by the formulae

$$x_j = a \cdot x' + b \cdot p' \qquad p_j = c \cdot x' + d \cdot p'$$

where (a, b) is the jth line of the matrix s and (c, d) its $(n+j)$th line. Writing $a = (a_1, \ldots, a_n)$, and so on, we have

$$\begin{aligned} (\Delta x_j)^2 &= \sum_{i=1}^{n} a_i^2 (\Delta x_i')^2 + b_i^2 (\Delta p_i')^2 \\ (\Delta p_j)^2 &= \sum_{i=1}^{n} c_i^2 (\Delta x_i')^2 + d_i^2 (\Delta p_i')^2. \end{aligned} \tag{12}$$

Setting

$$\alpha = (a_1 \Delta x_1, \ldots, a_n \Delta x_n) \qquad \beta = (b_1 \Delta p_1, \ldots, b_n \Delta p_n)$$
$$\gamma = (c_1 \Delta x_1, \ldots, c_n \Delta x_n) \qquad \delta = (d_1 \Delta p_1, \ldots, d_n \Delta p_n)$$

the formulae (12) can be written

$$(\Delta x_j)^2 = \alpha^2 + \beta^2 \qquad (\Delta p_j)^2 = \gamma^2 + \delta^2$$

and hence, by the Cauchy–Schwartz inequality,

$$(\Delta p_j)^2 (\Delta x_j)^2 \geqslant (\alpha \cdot \delta - \beta \cdot \gamma)^2.$$

Since we have, by definition of α, β, δ, γ:

$$\alpha \cdot \delta - \beta \cdot \gamma = \sum_{i=1}^{n} (a_i d_i - b_i c_i) \Delta p_i \Delta x_i$$

it follows that

$$\Delta p_j \Delta x_j \geqslant \left| \sum_{i=1}^{n} (a_i d_i - b_i c_i) \right| \varepsilon.$$

Now, the fact that the flow governing the time evolution of the system is Hamiltonian implies that the matrix in (11) is symplectic; now

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ symplectic} \iff \begin{cases} A^T C,\, D^T B \text{ symmetric} \\ A^T D - C^T B = I \end{cases}$$

and the condition $A^T D - C^T B = I$ implies that we must have

$$\sum_{i=1}^{n} a_i d_i - b_i c_i = 1 \tag{13}$$

which concludes the proof. □

Remark 13. The last step (formula (13)) in the proof of the proposition is essential, because it is there were the fact that the motion is Hamiltonian intervenes: if the flow consisted of just volume-preserving diffeomorphisms, the quantity

$$a \cdot d^T - b \cdot c^T = \sum_{i=1}^{n} a_i d_i - b_i c_i$$

could priori take any value (even zero). It turns out that the identity formula (13) can be used to give a proof of the non-squeezing theorem in the linear case (see [14]).

6. Conclusion... or beginning?

Gromov's theorem is an example of the deep and rich physical results that can be obtained by a careful analysis of the symplectic structure underlying mechanics; it is in that sense a perfect illustration of what Gotay and Isenberg [10] call the 'symplectization of Science'; also see Tuynman's interesting paper [22] on 'prequantization'. From a purely physical point of view, the theory we have sketched is semi-classical (it leads to the Keller–Maslov quantum condition, which is known to hold only in the limit of large quantum numbers; see [3] for numerous examples). It can be used to justify the axiomatic presentation we have given of semi-classical mechanics in [7,8]. It would be very interesting to investigate whether Gromov's theorem (or related topological properties) could be used to obtain a better understanding of the following points:

- As pointed out in remark 11, the passage from the ground-state condition (9) to the Keller–Maslov quantization condition (10) for excited states requires an extraneous hypothesis (either a purely physical assumption, Planck's law or a mathematical assumption), the single-valuedness of some 'phase-space wavefunctions'. Perhaps a modification of our basic postulates (1)–(2), or the addition of another 'topological' postulate could allow one to recover the excited states, too.
- The Keller–Maslov quantization condition is only exact for physical systems with quadratic Hamiltonian functions (see for instance [3, 15, 16]). It would be interesting to investigate whether there is some kind of 'quantum Gromov theorem' allowing us to retrieve the Keller–Maslov condition (10) without invoking any WKB argument. This would probably also shed some light on the quantization of non-integrable systems. The first step towards such a theory could very well lie in a restatement of postulate 1 in terms of 'symplectic cells' in the group $Sp(n)$ together with a study of the 'monodromy' of the orbits.

Let us conclude by noting that our results indicate that even classical (Hamiltonian) mechanics can have some of the characteristics of quantum mechanics. In particular, proposition 12 shows that one should be careful when using classical well known 'obvious' results such as Liouville's theorem, in particular when dealing with classical chaos. It is likely that the principle of the symplectic camel introduces limitations in chaotic behaviour, similar to those imposed by standard quantum mechanics. It would perhaps be interesting to reexamine Berry's discussion [2] of the motion of hyperion from this point of view.

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